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VALIDITY OF THE MULTIPLE-SCALE SOLUTION FOR A SUBHARMONIC RESONANCE RESPONSE OF A BAR WITH A NON-LINEAR BOUNDARY CONDITION

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In order to examine the validity of an asymptotic solution obtained from the method of multiple scales, we investigate a third order subharmonic resonance response of a bar constrained by a non-linear spring to a harmonic excitation. The motion of the bar is governed by a linear partial differential equation with a non-linear boundary condition. The non-linear boundary and initial value problem is solved by using the finite difference method. The numerical solution is compared with the asymptotic solution.

1. INTRODUCTION

In order to analyze non-linear vibrations of structural elements, many have used the method of multiple scales, which has been known to give a uniformly valid approximation as long as a specific system parameter is small. However, we cannot rely fully on the approximation, because there is no criterion on how small the parameter should be. Thus checking numerically and/or experimentally the validity of the approximation is essential, especially when the approximation disagrees with our intuition.

For instances, according to Nayfeh and Asfar [1], Hadian and Nayfeh [2] and Lee and Kim [3], secondary resonance responses can always be excited for all large values of the frequency detuning parameter, but "physically this is not the case", as stated by Nayfeh and Asfar [1]. The reason for the statement is as follows. The increase in the parameter causes the excitation frequency to meet another natural frequency, and then the system is governed by a primary resonance corresponding to the natural frequency rather than the secondary resonance. The analysis of the primary resonance starts with a different assumption on the magnitude of the excitation amplitude from the case of the secondary resonance. Thus we have to abandon the approximation for the secondary resonance when the parameter escapes from some range of the parameter, which the analysis does not tell us. Eventually, we have to rely on the numerical and/or experimental means to estimate the range.

In this study, to check the validity of the approximate responses for the secondary resonance we examine the longitudinal response of a bar with a non-linear boundary condition as in reference [1]. One end of the bar is clamped and the other end is constrained by a non-linear spring to a harmonic excitation. The finite difference method is used to solve the non-linear problem given by a linear partial differential equation and a non-linear boundary condition. The numerical solution is compared with the approximate solution by Nayfeh and Asfar [1].

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2. EQUATIONS OF MOTION AND STEADY STATE RESONANCE RESPONSES

We consider a system which consists of a bar and a spring, as shown in Figure 1. The right end of the bar is constrained by a non-linear spring and is subjected to harmonic excitation. The governing equation of motion of the structural system is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\varepsilon\mu(x)\frac{\partial u}{\partial t}, \tag{1}$$

$$u = 0 \quad \text{at } x = 0, \qquad \frac{\partial u}{\partial x} + \alpha u + \varepsilon u^3 = 2F\cos\Omega t \quad \text{at } x = 1. \tag{2, 3}$$

Using the method of multiple scales, Nayfeh and Asfar [1] obtained a uniform first order expansion of the solution for the third order subharmonic resonance ($\Omega = 3\omega_n + \sigma$, where $\sigma = \varepsilon\hat{\sigma}$ is a detuning parameter and $\hat{\sigma} = O(1)$) as follows:

$$u(x, t) = u_{sr}(x, t) + u_{nr}(x, t) + O(\varepsilon) \tag{4}$$

$$u_{sr}(x, t) = a_n G(x, \omega_n)\cos\left(\tfrac{1}{3}\Omega t - \tfrac{1}{3}\gamma\right), \qquad u_{nr}(x, t) = 2\Lambda G(x, \Omega)\cos\Omega t, \tag{5, 6}$$

where

$$G(x, \omega) = \frac{\sin\omega x}{\sin\omega}, \qquad \Lambda = \frac{F\sin\Omega}{\Omega\cos\Omega + \alpha\sin\Omega} \tag{7, 8}$$

The natural frequencies ω_n, poles of Λ, are given by the following characteristic equation.

$$\omega\cos\omega + \alpha\sin\omega = 0. \tag{9}$$

The nth mode amplitude a_n of the deflection u_{sr} is given by the steady state ($a_n' = 0$, $\gamma' = 0$) of the system of autonomous ordinary differential equations

$$a_n' = -\mu_n a_n - \frac{3\Gamma_n\Lambda}{4\omega_n} a_n^2 \sin\gamma, \tag{10}$$

$$a_n\gamma' = \hat{\sigma}a_n - \frac{9\Gamma_n}{8\omega_n} a_n^3 - \frac{9\Gamma_n\Lambda^2}{\omega_n} a_n - \frac{9\Gamma_n}{4\omega_n} a_n \sum_{r\neq n}^{\infty} a_r^2 - \frac{9\Gamma_n\Lambda}{4\omega_n} a_n^2\cos\gamma, \tag{11}$$

where μ_n and Γ_n are given by

$$\mu_n = \Gamma_n(\sin^2\omega_n)^{-1}\int_0^1 \mu(x)\sin^2\omega_n x\,\mathrm{d}x, \tag{12}$$

$$\Gamma_n = 4\omega_n\sin^2\omega_n(2\omega_n - \sin 2\omega_n)^{-1}. \tag{13}$$

Since the deflection u_{sr} is due to the subharmonic resonance response (a_n and γ), it is called the subharmonic resonance deflection. The deflection is similar to the vibration of the nth natural mode because it has the frequency near ω_n and natural mode shape $G(x, \omega_n)$. On the other hand, the deflection u_{nr} has the mode shape $G(x, \Omega)$ and the same frequency as

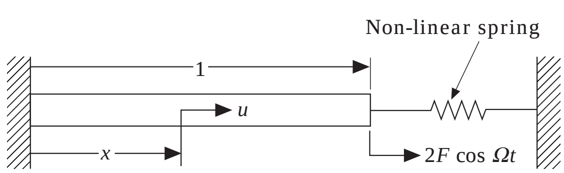

Figure 1. A schematic diagram of a bar with a non-linear boundary condition.

the excitation frequency Ω, which is not close to the natural frequency ω_n. The amplitude of u_{nr} is governed by Λ, which is not quite large because Ω is not close to ω_n. Now the deflection u_{nr} is called the non-resonance deflection because it is due to the non-resonance response Λ. By the observation of equations (10) and (11) we can see that there are two types of steady state responses ($a_n' = \gamma' = 0$) such as (I) $a_n = 0$ and (II) $a_n \neq 0$. For the steady state response of $a_n = 0$ this non-resonance deflection becomes the deflection $u(x, t)$. In other words, this non-resonance deflection is the only deflection that we can obtain in linear analysis, because this deflection is due to the response of the forced vibration of the linear system which has the same frequency as the excitation frequency.

3. FINITE DIFFERENCE ANALYSIS

In order to use a finite difference procedure [4, 5] for solving the problem, we replace the continuous problem domain ($0 \leqslant x \leqslant 1, 0 \leqslant t < \infty$) by a finite difference mesh or grid, as shown in Figure 2.

To reduce equations (1)–(3) to difference equations, we let

$$u_i^j = u(x, t) = u(x_0 + i\Delta x, t_0 + j\Delta t), \tag{14}$$

$$x = x_0 + i\Delta t \qquad (i = 0, 1, 2, \ldots, N), \tag{15}$$

$$t = t_0 + j\Delta t \qquad (j = 0, 1, 2, \ldots, M). \tag{16}$$

Developing Taylor series expansions for u_i^j and u_i^{j-2} about u_i^{j-1} gives

$$u_i^j = u_i^{j-1} + \Delta t \frac{\partial u}{\partial t}\bigg|_i^{j-1} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\bigg|_i^{j-1} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\bigg|_i^{j-1} + \frac{(\Delta t)^4}{4!}\frac{\partial^4 u}{\partial t^4}\bigg|_i^{j-1} + \ldots, \tag{17}$$

$$u_i^{j-2} = u_i^{j-1} - \Delta t \frac{\partial u}{\partial t}\bigg|_i^{j-1} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\bigg|_i^{j-1} - \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\bigg|_i^{j-1} + \frac{(\Delta t)^4}{4!}\frac{\partial^4 u}{\partial t^4}\bigg|_i^{j-1} + \ldots. \tag{18}$$

Using equations (17) and (18), we can reduce terms in equation (1) to the central difference representations as follows:

$$\frac{\partial^2 u}{\partial t^2}\bigg|_i^{j-1} = \frac{u_i^j - 2u_i^{j-1} + u_i^{j-2}}{(\Delta t)^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}\bigg|_i^{j-1} + \ldots$$

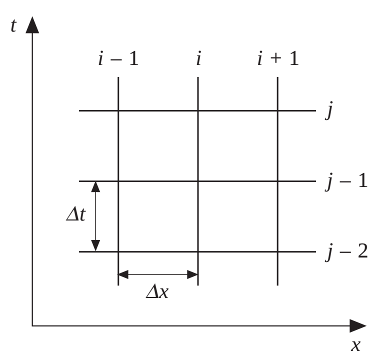

Figure 2. A schematic diagram of finite difference.

$$= \frac{u_i^j - 2u_i^{j-1} + u_i^{j-2}}{(\Delta t)^2} + O[(\Delta t)^2], \tag{19}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{j-1} = \frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{(\Delta x)^2} + O[(\Delta x)^2], \tag{20}$$

$$\left.\frac{\partial u}{\partial t}\right|_i^{j-1} = \frac{u_i^j - u_i^{j-2}}{2\Delta t} - \frac{(\Delta t)^2}{6}\left.\frac{\partial^3 u}{\partial t^3}\right|_i^{j-1} + \ldots = \frac{u_i^j - u_i^{j-2}}{2\Delta t} + O[(\Delta t)^2]. \tag{21}$$

Using these representations we can reduce equation (1) to a difference equation with a truncation error of $O[(\Delta x)^2, (\Delta t)^2]$ as follows:

$$\frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{(\Delta x)^2} = \frac{u_i^j - 2u_i^{j-1} + u_i^{j-2}}{(\Delta t)^2} + 2\varepsilon\mu\,\frac{u_i^j - u_i^{j-2}}{2\Delta t}$$

$$(i = 1, 2, \ldots, N-1,\ j = 2, 3, \ldots, M). \tag{22}$$

In a bench test for the free vibration of a fixed–free bar ($\varepsilon = \mu = \alpha = F = 0$ in equations (1)–(3)) we obtained a faithful result by using the third-order backward difference representation for $\partial u/\partial x$ in equation (3).

Developing Taylor series expansions for u_{N-1}^j, u_{N-2}^j and u_{N-3}^j about u_N^j gives

$$u_{N-1}^j = u_N^j - \Delta x\left.\frac{\partial u}{\partial x}\right|_N^j + \frac{(\Delta x)^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_N^j - \frac{(\Delta x)^3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_N^j + \frac{(\Delta x)^4}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_N^j + \ldots, \tag{23}$$

$$u_{N-2}^j = u_N^j - 2\Delta x\left.\frac{\partial u}{\partial x}\right|_N^j + \frac{(2\Delta x)^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_N^j - \frac{(2\Delta x)^3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_N^j + \frac{(2\Delta x)^4}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_N^j + \ldots, \tag{24}$$

$$u_{N-3}^j = u_N^j - 3\Delta x\left.\frac{\partial u}{\partial x}\right|_N^j + \frac{(3\Delta x)^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_N^j - \frac{(3\Delta x)^3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_N^j + \frac{(3\Delta x)^4}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_N^j + \ldots. \tag{25}$$

Eliminating the terms with Δx from equations (23) and (24), we have

$$u_{N-2}^j - 2u_{N-1}^j = -u_N^j + (\Delta x)^2\left.\frac{\partial^2 u}{\partial t^2}\right|_N^j - (\Delta x)^3\left.\frac{\partial^3 u}{\partial t^3}\right|_N^j + \frac{7(\Delta x)^4}{12}\left.\frac{\partial^4 u}{\partial t^4}\right|_N^j + \ldots. \tag{26}$$

Eliminating the terms with $(\Delta x)^2$ from equations (25) and (26), we have

$$\left.\frac{\partial u}{\partial x}\right|_N^j = \frac{11u_N^j - 18u_{N-1}^j + 9u_{N-2}^j - 2u_{N-3}^j}{6\Delta x} - \frac{5(\Delta x)^3}{8}\left.\frac{\partial^4 u}{\partial x^4}\right|_N^j + \ldots$$

$$= \frac{11u_N^j - 18u_{N-1}^j + 9u_{N-2}^j - 2u_{N-3}^j}{6\Delta x} + O[(\Delta x)^3]. \tag{27}$$

Thus, equations (2) and (3) are reduced to the following difference equations:

$$u_0^j = 0, \qquad \frac{11u_N^j - 18u_{N-1}^j + 9u_{N-2}^j - 2u_{N-3}^j}{6\Delta x} + \alpha u_N^j + \varepsilon(u_N^j)^3 = 2F\cos[\Omega(t_0 + j\Delta t)].$$

(28, 29)

In the end, the boundary and initial value problem (1)–(3) is reduced to the difference equations (22), (28) and (29), from which we can obtain numerical solutions through the four steps, as follows.

Step 1. Substitute the initial and boundary conditions for $t = t_0$ into equations (28) and (29).

Step 2. Use the explicit method to solve equation (22) for u_i^j $(i = 1, 2, \ldots, N-1)$.

Step 3. Substitute the above u_i^j into equation (29) to obtain u_N^j.

Step 4. Iterate steps 2 and 3 to obtain u_i^j $(i = 1, 2, \ldots, N)$ and $j = 1, 2, \ldots, M)$.

For this study we take $\Delta t = (2\pi/\Omega)/3000$ and $\Delta x = 0{\cdot}01$ $(N = 100)$.

4. NUMERICAL RESULTS

In this study we consider the case of $\Omega \approx 3\omega_1$ $(n = 1)$ and $\{\varepsilon, \mu_1, \alpha\} = \{0{\cdot}01, 0{\cdot}1, 0{\cdot}3\}$. The natural frequencies are $\{\omega_1, \omega_2, \omega_3, \ldots\} = \{1{\cdot}7414,\ 4{\cdot}7751,\ 7{\cdot}8920, \ldots\}$.

Using equations (10) and (11) and stability criteria, we have plotted the amplitude parameter (σ and F) response curves in Figures 3 and 4, where solid and dotted lines denote, respectively, stable and unstable responses. There exists one pair of non-zero amplitude responses. It has the stable and unstable branches. Since the zero amplitude response is stable, the system can have two stable steady state periodic responses. In this case, the long-term response of the system depends on the initial condition. The symbols △ and ○ obtained by finite difference analysis denote, respectively, the zero amplitude and non-zero amplitude responses. One of the difficulties in obtaining numerical solutions of the boundary and initial value problem is to choose proper initial conditions. Each of these initial conditions implies 200 numbers (one velocity and one displacement at each point i of the bar). In this study, for convenience, we use the stable solutions (solid lines in the figures) obtained analytically to choose proper initial conditions.

In Figure 3 are shown three saddle-node bifurcation points, σ_A, σ_C and σ_D. Because of the zero amplitude response, a jump phenomenon occurs at each of these bifurcation points. The figure shows that the amplitude of the stable non-zero amplitude response increases with the detuning parameter σ. However, we can easily expect that this result is physically invalid. As σ increases and the excitation frequency Ω increases up to ω_3

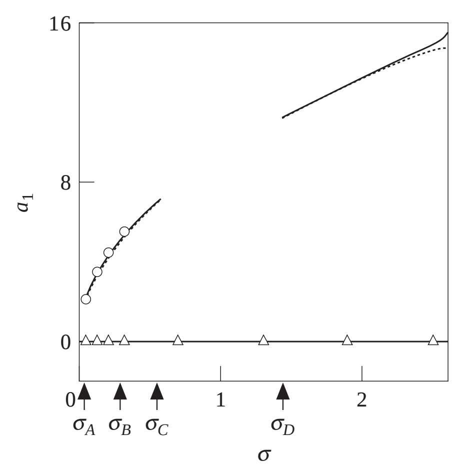

Figure 3. Variation of the amplitude of subharmonic responses with the detuning parameter σ: $F = 0{\cdot}2$, $\varepsilon = 0{\cdot}01$, $\mu_1 = 0{\cdot}1$, $\alpha = 0{\cdot}3$ and $\omega_1 = 1{\cdot}7414$. —, Stable; · · · ·, unstable. Numerical solutions: ○, non-zero amplitude resonance solution; △, zero amplitude resonance solution.

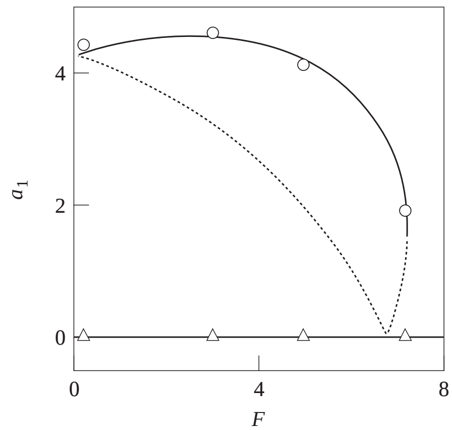

Figure 4. Variation of the amplitude of subharmonic responses with the amplitude of the excitation F: $\sigma = 0{\cdot}21$, $\varepsilon = 0{\cdot}01$, $\mu_1 = 0{\cdot}1$, $\alpha = 0{\cdot}3$, $\omega_1 = 1{\cdot}7414$. —, Stable; · · · · , unstable. Numerical solutions: ○, non-zero amplitude resonance solution; △, zero amplitude resonance solution.

($\sigma = 2{\cdot}668$), the effect of the subharmonic resonance should disappear. The result from the finite difference analysis shows that the approximate solution obtained using the method of multiple scales is valid only for a very limited region of σ ($\sigma_A < \sigma < \sigma_B$), as expected. Of course, the invalidity of the solution for large σ may not be so crucial because it is well known that the effect of the resonance is meaningful in a limited region of σ. However, this result is in a marked contrast to the cases of the primary resonance [6, 7] where the first order approximations expect very well that the non-zero amplitude resonance responses exist for a limited region of σ.

In Figure 4 it is shown that the amplitude of the stable non-zero amplitude response decreases with the excitation amplitude F. This phenomenon can appear only in the secondary resonances. Unlike Figure 3, Figure 4 shows that the approximate solution agrees very well with the numerical solutions. Integrating equations (1)–(3) numerically, we can draw Figures 5 and 6 to show the time histories of the steady state deflection at the right end of the bar ($x = 1$). These figures correspond to the non-zero amplitude and

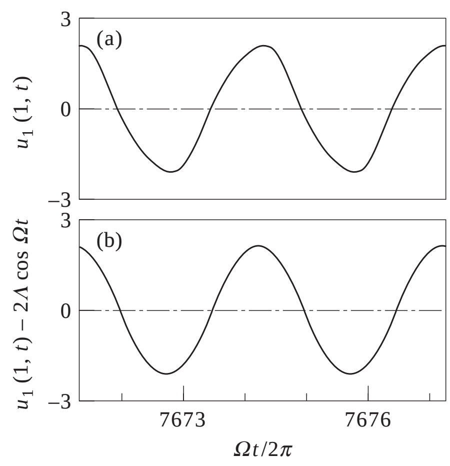

Figure 5. Time histories corresponding to the steady state stable deflection with non-zero resonance amplitude: $\sigma = 0{\cdot}21$, $F = 0{\cdot}2$, $\varepsilon = 0.01$, $\mu_1 = 0{\cdot}1$, $\alpha = 0{\cdot}3$ and $\omega_1 = 1{\cdot}7414$. (a) Total deflection; (b) resonance deflection.

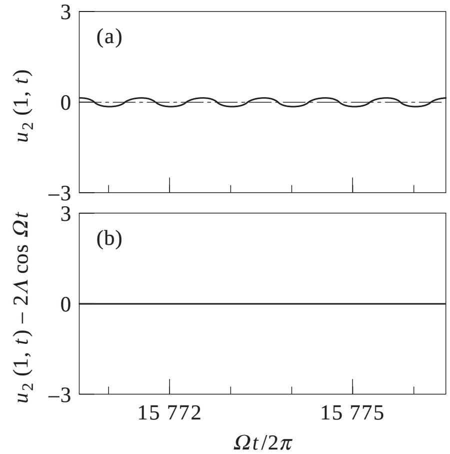

Figure 6. Time histories corresponding to the steady state stable deflection with zero resonance amplitude: $\sigma = 0{\cdot}21$, $F = 0{\cdot}2$, $\varepsilon = 0.01$, $\mu_1 = 0{\cdot}1$, $\alpha = 0{\cdot}3$ and $\omega_1 = 1{\cdot}7414$. (a) Total deflection; (b) resonance deflection.

zero amplitude responses, respectively. We used the different initial conditions to obtain the numerical solutions in Figures 5(a) and 6(a). Subtracting the non-resonance response, we can obtain the non-zero amplitude and zero amplitude responses shown in Figures 5(b) and 6(b), respectively. The amplitudes obtained from Figures 5(b) and 6(b) are denoted, respectively, by ○ and △ in Figures 3 and 4. The periods of the deflections shown in Figures 5(b) and 6(a) are three times and one time the excitation period, respectively. This shows that the approximate solution in equations (5) and (6) estimates the periods as well as the amplitudes of the responses.

5. CONCLUSIONS

In order to examine the validity of the asymptotic solution for the subharmonic resonance response, we consider a bar constrained by a non-linear spring to a harmonic excitation. The approximate solution shows that the third order subharmonic resonance response can be excited for large values of the frequency detuning parameter. However, the numerical solution obtained by the finite difference analysis shows that the approximate solution is valid only within a very limited range of the parameter.

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